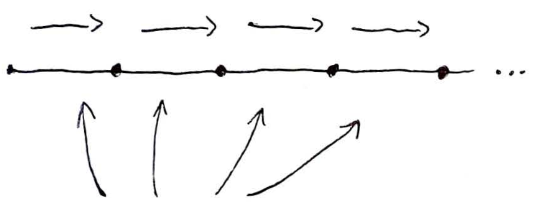
Zadanie 14.c Wylicz funkcję tworzącą ciągu $a_n = \binom{n+k}{k}$

$$A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n \geq 0} \underbrace{\binom{n+k}{k}}_{\substack{\text{liczba podziałów} \\ \text{liczby } n \text{ na } k+1 \\ \text{składników (istotna} \\ \text{jest ich kolejność)}}} x^n = \left(\sum_{m\geq 0} x^m\right)^{k+1} = \left(\frac{1}{1-x}\right)^{k+1}$$

Zadanie 10.



z par p przyjęta wartość jest negowana

p_n - prawdopodobieństwo uzyskania oryginału po n transmisjach

$p_0 = 1$, $p_1 = 1-p$

$$p_n = \underbrace{p_{n-1}\cdot(1-p)}_{\substack{\text{nie doszło} \\ \text{do błędu,} \\ \text{wartość nie} \\ \text{została zmieniona}}} + \underbrace{(1-p_{n-1})\cdot p}_{\substack{\text{do poprzedniego punktu} \\ \text{dotarła zmieniona} \\ \text{wartość}}}$$

$$p_n = (1-2p)\cdot p_{n-1} + p$$

$E\langle p_n\rangle = (1-2p)p_n + p$ ← ciąg po przesunięciu, czyli $\langle a_0, a_1, a_2, a_3, \dots\rangle \xrightarrow{E} \langle a_1, a_2, a_3, \dots\rangle$

$$(E-(1-2p))\langle p_n\rangle = p$$

$$(E-1)(E-(1-2p))\langle p_n\rangle = 0$$

1° $p \neq 0 \Rightarrow p_n = \alpha(1-2p)^n + \beta\cdot 1^n$

$$- \begin{cases} p_0 = 1 = \alpha + \beta \\ p_1 = 1-p = \alpha(1-2p)+\beta \end{cases}$$

$$p = \alpha\cdot 2p \Rightarrow \alpha = \frac{1}{2}, \beta = \frac{1}{2}$$

$$p_n = \frac{1}{2}\left(\left(\frac{1}{2}\right)^n + 1\right)$$

2° ...

$p_n = 1$

Zadanie 7.

$x_0 = 1$

$x_1 = 25$

$x_2 = 26 + 24 \cdot 25 = 1 + 25^2$

ciągi długości $n-1$ o parzystej liczbie "a"

$$x_n = 1 \cdot (\underbrace{26^{n-1} - x_{n-1}}_{\text{ciągi długości } n-1 \text{ o nieparzystej liczbie "a"}}) + 25 \cdot x_{n-1} =$$

"a" na końcu

"a" nie jest na końcu

$$= 26^{n-1} + 24 x_{n-1}$$

$$x_{n+1} - 24 x_n = 26^n$$

$$(E-24)\langle x_n \rangle = 26^n$$

$$(E-24)(E-26)\langle x_n \rangle = 0$$

$$x_n = \alpha \cdot 24^n + \beta \cdot 26^n$$

$$\begin{cases} \alpha + \beta = 1 & (x_0) \\ 24\alpha + 26\beta = 25 & (x_1) \end{cases} \Rightarrow \begin{cases} \alpha = \frac{1}{2} \\ \beta = \frac{1}{2} \end{cases}$$

Zadanie 8. $s_n = \sum_{i=1}^{n} i \cdot 2^i$, $s_n = s_{n-1} + n \cdot 2^n$

$$(E-1)\langle s_n \rangle = (n+1) \cdot 2^{n+1}$$

$$(E-2)\langle (n+1)2^{n+1} \rangle = \overbrace{(n+2) \cdot 2^{n+2}}^{E\langle a_n \rangle} \overbrace{- 2 \cdot (n+1) 2^{n+1}}^{-2\langle a_n \rangle}$$

$$= 2^{n+2}$$

$$(E-1)(E-2)^2\langle s_n \rangle = 0$$

$$s_n = \alpha \cdot 2^n + \beta \cdot n \cdot 2^n + \gamma \cdot 1^n$$

$$f(x) = \sum_{i=1}^{n} i \cdot x^i$$

Zadanie 12. c

$$\frac{1}{(k+1)(k+2)(k+3)} = \frac{\alpha}{k+1} + \frac{\beta}{k+2} + \frac{\gamma}{k+3} \quad / \cdot (k+1)(k+2)(k+3)$$

$$1 = \alpha(k+2)(k+3) + \beta(k+1)(k+3) + \gamma(k+1)(k+2)$$

Podstawiając $k=-1$, $k=-2$, $k=-3$ dostajemy:

1° $k=-1$ $\quad \alpha \cdot 1 \cdot 2 = 1 \quad \Rightarrow \alpha = \frac{1}{2}$

2° $k=-2$ $\quad \beta \cdot (-1) \cdot 1 = 1 \quad \Rightarrow \beta = -1$

3° $k=-3$ $\quad \gamma \cdot (-2) \cdot (-1) = 1 \quad \Rightarrow \gamma = \frac{1}{2}$

$$\sum_{k=1}^{n} \left(\frac{\frac{1}{2}}{k+1} - \frac{1}{k+2} + \frac{\frac{1}{2}}{k+3} \right) =$$

$$= \frac{\frac{1}{2}}{2} - \frac{1}{3} + \frac{\frac{1}{2}}{4} + \frac{\frac{1}{2}}{3} - \frac{1}{4} + \frac{\frac{1}{2}}{5} + \frac{\frac{1}{2}}{4} - \frac{1}{5} + \frac{\frac{1}{2}}{6} + \dots =$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3}\left(\frac{1}{2} - 1\right) + \frac{1}{n+3} \cdot \frac{1}{2} + \frac{1}{n+2}\left(\frac{1}{2} - 1\right)$$

Zadanie 13. c $\qquad s_n = a_0 + a_1 + a_2 + \dots + a_n$

$$\sum_{n \geq 0} s_n x^n = \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k \right) x^n = \sum_{n \geq 0} \left(\sum_{k=0}^{n} (a_k x^k) \cdot x^{n-k} \right) =$$

$$= \left(\sum_{n \geq 0} a_n x^n \right) \left(\sum_{n \geq 0} x^n \right) = A(x) \cdot \frac{1}{1-x}$$

Zadanie 13.6 $\quad c_n = \frac{a_n}{n}, \quad c_0 = 0$

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

$$C(x) = \sum_{n \geq 1} c_n x^n = \sum_{n \geq 1} \frac{a_n x^n}{n} = \sum_{n \geq 1} \int a_n x^{n-1} dx =$$

$$= \int \sum_{n \geq 1} a_n x^{n-1} dx = \int \frac{A(x) - A(0)}{x} dx$$

Zadanie 13. d

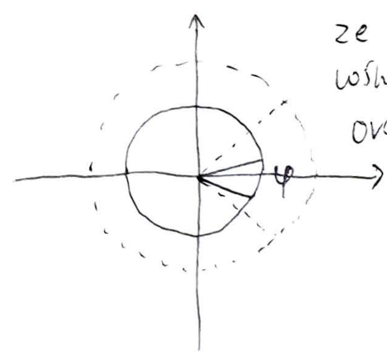
$$d_n = \begin{cases} a_n & : n = 2k \\ 0 & : n = 2k+1 \end{cases} \qquad D(x) = \left(A(x) + A(-x)\right)/2$$

$$2 \nmid n : \quad a_n\left(x^n + (-x)^n\right) = 0$$

$$2 \mid n : \quad a_n\left(x^n + (-x)^n\right) = 2a_n \cdot x^n$$

Zadanie (egzamin) $\quad x_n = 4x_{n-1} - (4 + \varepsilon^2) \cdot x_{n-2}, \quad x_0 = 1, \; x_1 = 2$

Wykaż, że $\forall \varepsilon > 0 \; \exists k \in \mathbb{N} \; x_k < 0$



ze wzrostem n rośnie promień oraz kąt

$$x_n = \frac{1}{2}\left((2 + \varepsilon i)^n + (2 - \varepsilon i)^n\right)$$