Zadanie 1.15. $f(x)=(n+1)x-\lfloor nx\rfloor=c$

1° Jeśli $f(x)=c$, to $\{c\}=\{f(x)\}=\{f(\{x\})\}$.

2° W takim razie każde rozw. $f(x)=c$ jest postaci $x=z+r$, $z\in\mathbb{Z}$, $r\in[0,1)$ oraz $\{f(r)\}=\{c\}$. Ponadto, dla każdego r istnieje dokładnie jedno $z\in\mathbb{Z}$ takie, że $f(z+r)=c$.

Ad. 1: $\{f(x)\}=\{(n+1)x\}=\{(n+1)\{x\}\}=\{c\}$

Ad. 2: $f(x)=x+nx-\lfloor nx\rfloor=x+\{nx\}=\lfloor x\rfloor+\{x\}+\{n\{x\}\}=c$

$$z+r+\{nr\}=c$$

Ile rozwiązań w $[0,1)$ ma równanie $\{(n+1)x\}=\{c\}$?

$(n+1)x=\{c\}+k$ dla pewnego $k\in\mathbb{Z}$

$x=\frac{\{c\}+k}{n+1}$ dla $k\in\{0,1,\dots,n\}$, bo $x\in[0,1)$.

Zadanie 1.12. $\left(n+2+O\left(\frac{1}{n}\right)\right)^n=n^ne^2\left(1+O\left(\frac{1}{n}\right)\right)$ $\quad /:n^n$

$$\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)^n=e^2\left(1+O\left(\frac{1}{n}\right)\right)$$

$1+x\le e^x$

$\le$: $\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)^n\le\left[\left(1+\frac{2}{n}\right)\left(1+O\left(\frac{1}{n^2}\right)\right)\right]^n=\left(1+\frac{2}{n}\right)^n\cdot\left(1+O\left(\frac{1}{n^2}\right)\right)^n\le\left(e^{\frac{2}{n}}\right)^n\cdot\left(e^{O\left(\frac{1}{n^2}\right)}\right)^n=$

$=e^2\cdot e^{O\left(\frac{1}{n}\right)}\le e^2\cdot\left(1+O\left(\frac{1}{n}\right)\right)$

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$$\boxed{\left(1+\frac{1}{n}\right)^n\le e\le\left(1+\frac{1}{n}\right)^{n+1}}$$

$\ge$: $e^2\left(1+O\left(\frac{1}{n}\right)\right)\le\cancel{\left(1+\frac{2}{n}\right)^n\cdot O\left(1+\frac{1}{n}\right)}$

$\le\left(\left(1+\frac{2}{n}\right)^{\frac{n}{2}+1}\right)^2\cdot\left(1+O\left(\frac{1}{n}\right)\right)=\left(1+\frac{2}{n}\right)^n\cdot\left(1+\frac{2}{n}\right)^2\cdot\left(1+O\left(\frac{1}{n}\right)\right)\le$

$\le\left(1+\frac{2}{n}\right)^n\cdot\left(1+O\left(\frac{1}{n}\right)\right)\le\left(1+\frac{2}{n}\right)^n\cdot e^{O\left(\frac{1}{n}\right)}\le\left(1+\frac{2}{n}\right)^n\cdot\left(1+O\left(\frac{1}{n^2}\right)\right)^{n+1}\le$

$\le\left(1+\frac{2}{n}\right)^n\cdot\left(1+O\left(\frac{1}{n^2}\right)\right)^n\le\left[\left(1+\frac{2}{n}\right)\left(1+O\left(\frac{1}{n^2}\right)\right)\right]^n=\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)^n$

Zadanie 2.12. Podział prostopadłościanu na kawałki za pomocą n cięć.

$\mathbf{R}(d,n)$ - max liczba obszarów na które n hiperpłaszczyzn (tj. prostych dla $d=2$, płaszczyzn dla $d=3$), d - liczba wymiarów, na które można podzielić $\mathbb{R}^d$

$$R(3,n) \leqslant R(3,n-1) + R(2,n-1)$$

Uogólnione dla dowolnego d,n: $R(d,n) \leq R(d,n-1) + R(d-1,n-1)$, przyjmujemy $R(0,n) \equiv 1$ (podział punktu), dostajemy $R(1,0)=1$, $R(1,1)=2, \dots,$ $R(1,n)=n+1$, $R(2,0)=1$, $R(2,1)=2, \dots, R(2,n) = \binom{n}{2}+\binom{n}{1}+\binom{n}{0} = \frac{n^2+n}{2}+1$.

$$R(d,n) = \sum_{i=0}^{d} \binom{n}{1}; \quad R(d,n) = R(d,n-1)+R(d-1,n-1) =$$

$$= \sum_{i=0}^{d}\binom{n-1}{i} + \sum_{i=0}^{d-1}\binom{n-1}{i} =$$

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

$$= 1 + \sum_{i=0}^{d-1}\binom{n-1}{i+1} + \sum_{i=0}^{d-1}\binom{n-1}{i} =$$

$$= 1 + \sum_{i=0}^{d-1}\binom{n}{i+1} = \binom{n}{0} + \sum_{i=1}^{d}\binom{n}{i} = \sum_{i=0}^{d}\binom{n}{i}$$

Zadanie 2.9.

(a) $f(1)=1$, $f(n) = f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + 1$

n	1	2	3	4	5	6
$f(n)$	1	3	5	7	9	11

..., więc $f(n) \stackrel{?}{=} (2n-1)$

$$f(n) = f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + f\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + 1 = 2\left\lfloor \tfrac{n}{2} \right\rfloor - 1 + 2\left\lceil \tfrac{n}{2} \right\rceil - 1 + 1 =$$

$$= 2\left(\left\lfloor \tfrac{n}{2} \right\rfloor + \left\lceil \tfrac{n}{2} \right\rceil\right) - 1 = 2n-1$$

(b) $g(n) = g(\lfloor \frac{n}{2} \rfloor) + \lfloor \log n \rfloor = \lfloor \log n \rfloor + \lfloor \log \lfloor \frac{n}{2} \rfloor \rfloor + g\left(\left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \right\rfloor\right) =$

$= \lfloor \log n \rfloor + (\lfloor \log n \rfloor - 1) + g(\lfloor \frac{n}{4} \rfloor) =$

$= \lfloor \log n \rfloor + (\lfloor \log n \rfloor - 1) + \ldots + (\lfloor \log n \rfloor - \lfloor \log n \rfloor) + g(0) =$

$= \binom{\lfloor \log n \rfloor}{2}$

Zadanie 7.

(a) $a_0 = 1$, $a_{n+1} = (n+1)a_n + 1 \longrightarrow A_n = \frac{a_n}{n!}$

więc $A_{n+1} = \frac{a_{n+1}}{(n+1)!} = \frac{a_n}{n!} + \frac{1}{(n+1)!} = A_n + \frac{1}{(n+1)!}$

(d) $d_0 = 1$, $d_1 = 2$, $nd_n = (n-2)! \cdot d_{n-1} \cdot d_{n-2}$ $\quad / \cdot (n-1)!$

$D_n = n!\, d_n = ((n-2)!\, d_{n-2}) \cdot ((n-1)!\, d_{n-1})$